PAIRWISE COMPARISON AND RANKING IN TOURNAMENTS

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1. Introduction. This paper is concerned with the following ranking problem: $n \geqq 3$ items are compared pairwise. The results of all comparisons can be summarized in a *preference matrix* $A = (a_{ij})$ where $a_{ij} = 1, 0$, or $\frac{1}{2}$, respectively, according as item i is preferred to j, item j is preferred to i or no preference is expressed between i and j, respectively. Which is the best method of ranking all items in the "order of their preferences" provided A is known?

In tournaments of chess, which represent a canonical model for the above situation, it is customary to rank in descending order of the *scores* $s_i = \sum_j a_{ij}$. Since, however, other ranking procedures have been proposed, e.g., by Wei-Kendall [3], the problem arises how to characterize the "goodness" of any such procedure. In this paper we give such a characterization in terms of the "underlying probability structure" and then exhibit a class of such structures for which the usual ranking procedure by *scores* s_i is optimal.

In order to keep what follows as intuitive as possible we shall from now on use the terminology referring to chess tournaments, i.e., "player" for "item," "game" for "comparison" and "won," "lost" or "drawn" for the possible results of any comparison.

2. The underlying probability structure and the correct ranking.

2.1. In probabilistic terms a tournament can be described as follows. The a_{ij}'s appearing in the matrix A are considered as random variables which take on the values 0, $\frac{1}{2}$, and 1. For the purposes of this paper we shall, however, exclude draws, i.e., we assume for all $i \neq j$

$$[a_{ij} = 1] \text{ with probability } p_{ij}, \tag{1}$$

$$[a_{ij} = 0] \text{ with probability } q_{ij} = p_{ji}, \tag{2}$$

and

$$p_{ij} + p_{ji} = 1. \tag{3}$$

The case where draws are permitted will be discussed in a forthcoming paper by Huber [2] where our results are extended to more general random variables a_{ij}.

If the results of all games are independent, then the *probability matrix* $P = (p_{ij})$ describes the complete underlying *probability structure* of the *preference matrix* A, whose elements can then be looked upon as follows: (i) a_{ij} for $i < j$

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are independent zero-one trials each with probability p_{ij}; (ii) a_{ij} for $i > j$ are related to the set of the above random variables (i) by $a_{ij} = 1 - a_{ji}$; (iii) $a_{ii} = \frac{1}{2}$, i.e., constant for all i.

2.2. Suppose now that the matrix P is completely known; which would then be the *"correct ranking"* of the players? The following principles seem feasible.

(a) Define $m_i = \min_{j \neq i} p_{ij}$ and rank in descending order of m_i.

(b) Define $p_i = \sum_j p_{ij}$ and rank in descending order of p_i.

(c) Define $p^{(N)} = P^{N-1} p$ and rank in descending order of $p_i^{(N)}$ where $p = (p_1, p_2, \cdots, p_n)$ as defined in (b), $p^{(N)} = (p_1^{(N)}, p_2^{(N)}, \cdots, p_n^{(N)})$.

Procedure (c) is derived from the method proposed by Wei-Kendall [2]. These authors, however, apply this iteration procedure to the matrix A, not to the matrix P.

There is no doubt how the "correct ranking" should be defined in the following special case:

$$p_{ij} = F(\theta_i - \theta_j), \tag{4}$$

where $\{\theta_i\}$ are parameters associated with the players and $F(t)$ is any symmetric distribution function. Obviously the players should then be ranked in descending order of the θ_i. It is also easy to check that this ranking satisfies all three principles (a), (b), and (c) mentioned above. (The reader may observe that the special Case (4) arises if the performance of player i is of the form $\theta_i + \Delta_i$ and i wins over j if $\theta_i + \Delta_i > \theta_j + \Delta_j$, the fluctuation terms Δ_i being independent and identically distributed random variables.)

3. Formulation of goodness of ranking procedures.

3.1. The problem of ranking can now be described as follows: "Try to hit the 'correct ranking' (defined in terms of P) if you know the outcome of the tournament, i.e., the matrix A."

Remark. We may always assume that there is only one "correct" ranking; if P should happen to admit two or more, we would arbitrarily distinguish one of them, say d, as best (e.g., by considering P as a limiting case of suitable neighboring matrices having d as unique correct ranking).

In order to formalize this problem we define: $d = (d_1, d_2, \cdots, d_n)$ = ranking vector, where $d_i = k$ means that player number i has been assigned to the kth place in the chosen ranking; $L(d, P)$ = loss arising if ranking d is chosen and P is the true underlying probability matrix; δ = ranking procedure assigning to each matrix A a ranking vector $\delta(A)$. δ may be randomized. The risk of a given ranking procedure δ is then

$$R(\delta, P) = E_P[L(\delta(A), P)], \tag{5}$$

where E_P = expectation if P is the true underlying probability matrix.

Different ranking procedures are then compared on the basis of their risk functions; in particular we have

DEFINITION. The ranking procedure δ is uniformly better than (dominates) δ' for the class $\mathcal{P}$ of probability matrices if

$$R(\delta, P) \leqq R(\delta', P) \text{ for all } P \in \mathcal{P}.$$

4. Invariance.

4.1. Whenever we talk about "invariance under permutations of the players" it seems intuitively clear what this expression means, namely, that a renumbering of the players should not matter. The following is a formalization of this intuitive notion. We define

4.1.1. $\mathcal{P} = \{P\}$: space of probability matrices. In the special case [(4), Section 2.2] where $p_{ij} = F(\theta_i - \theta_j)$ with fixed F, $\mathcal{P}$ is in 1–1 correspondence with $\Theta = \{\theta\}$: parameter space.

4.1.2. $\mathcal{A} = \{A\}$: space of tournament outcomes (sample space).

4.1.3. $\mathfrak{D} = \{d\}$: space of rankings (decision space).

4.2. Then the "permutation of players" σ operates as follows on these three spaces:

4.2.1. On $\mathcal{P}$: $P \xrightarrow{\sigma} P^\sigma$ defined by $p_{ij}^\sigma = p_{\sigma(i)\sigma(j)}$ and similarly in the special case on Θ: $\theta \xrightarrow{\sigma} \theta^\sigma$ defined by $\theta_i^\sigma = \theta_{\sigma(i)}$.

4.2.2. On $\mathcal{A}$: $A \xrightarrow{\sigma} A^\sigma$ defined by $a_{ij}^\sigma = a_{\sigma(i)\sigma(j)}$.

4.2.3. On $\mathfrak{D}$: $d \xrightarrow{\sigma} d^\sigma$ defined by $d_i^\sigma = d_{\sigma(i)}$.

We then define:

(i) $L(d, P)$ is invariant under σ if $L(d^\sigma, P^\sigma) = L(d, P)$.

(ii) $\delta(A)$ is invariant under σ if $\delta(A) = d \Leftrightarrow \delta(A^\sigma) = d^\sigma$.

4.3. *Technical remark.* Observe that $(P^\sigma)^\tau = Q$ is defined by $q_{ij} = p_{\sigma(\tau(i)),\sigma(\tau(j))}$. Hence we have $(P^\sigma)^\tau = P^{\sigma\tau}$ and similarly $(\theta^\sigma)^\tau = \theta^{\sigma\tau}$, $(A^\sigma)^\tau = A^{\sigma\tau}$, $(d^\sigma)^\tau = d^{\sigma\tau}$. Thus, the group of all permutations operates on all these spaces from the right.

5. Solution of the reduced ranking problem.

5.1. The reduced ranking problem is defined as follows: The class $\mathcal{P}_0$ consists of a given matrix P and all matrices P^σ obtained from P by permutations of the players, i.e., $p_{ij}^\sigma = p_{\sigma(i),\sigma(j)}$, where σ stands for some permutation of the integers 1 to n. In other words, $\mathcal{P}_0$ is the orbit of P under the group of all permutations of the players. The loss function is defined by

$$\begin{aligned} L(d, P) &= 0 \qquad \text{if } d \text{ is the correct ranking,} \\ &= 1 \qquad \text{otherwise.} \end{aligned}$$

Find $\delta(A)$ which among all permutation invariant procedures is uniformly best for $\mathcal{P}_0$ (i.e., maximizes the probability of hitting the correct ranking).

5.2. Without loss of generality we can assume that $(1, 2, \cdots, n)$ is the correct ranking under P [hence $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is the correct ranking for P^σ].

The probability $W_P(A)$ that the outcome of the tournament is A, P being the true underlying probability matrix, can then be computed as follows:

$$W_P(A) = \prod_{\substack{i<j \\ a_{ij}=1}} p_{ij} \prod_{\substack{i<j \\ a_{ij}=0}} p_{ji} = \prod_{i \neq j} (p_{ij})^{a_{ij}} = \prod_{i \neq j} (p_{ji})^{1-a_{ij}}$$

[using (3) and the convention $0^0 = 1$].

The maximum likelihood principle leads then to the following procedure.

Reduced Ranking Procedure. *Determine the permutation σ for which $W_{P^\sigma}(A)$ is maximum. Then rank according to the true ranking for P^σ, that is, take the ranking vector $d = (\sigma(1), \sigma(2), \cdots, \sigma(n))$. If the maximum is attained for more than one σ, then choose one of them in an invariant way (e.g., at random with equal probability) in order to make the procedure invariant under permutations.*

The proof that the reduced ranking procedure minimizes the risk and hence that it is a solution of the reduced ranking problem is similar to the first part of the proof for Theorem 2 in Section 7. In fact, up to formula (18) it is identical except for a change of notation (P instead of θ). In view of the special form of the loss function, (18) is however just another way of characterizing maximum likelihood procedures.

For later purposes we note that if $p_{ij} > 0$ for all i, j, then $W_P(A)$ can be expressed as follows:

$$[W_P(A)]^2 = \prod_{i \neq j} p_{ij} \, (p_{ij}/p_{ji})^{a_{ij}} = C^2(P) \prod_{i \neq j} (p_{ij}/p_{ji})^{a_{ij}} \tag{6}$$

or

$$W_P(A) = C(P) \exp \left\{ \tfrac{1}{2} \sum_{i,j} a_{ij} c_{ij} \right\} \tag{6'}$$

with $c_{ij} = \log(p_{ij}/p_{ji})$ and $C(P) > 0$ depending on P but not on A.

Remark. In Bayesian terminology the above maximum likelihood method amounts to choosing a ranking with maximum *a posteriori* probability given the outcome A, for the uniform *a priori* distribution on $\mathcal{P}_0$.

6. The general ranking problem.

6.1. In the previous section we solved the ranking problem for a very restricted class of probability matrices. In general, of course, we are interested in much larger classes. Still it seems reasonable to postulate, in the general case too, that the optimum ranking procedure be invariant under permutations of the players, since any procedure which does not have this property would not qualify as being fair.

Hence we define the general ranking problem: Given a class $\mathcal{P}$ and a loss function $L(d, P)$, find a ranking procedure $\delta(A)$ which for $\mathcal{P}$ is uniformly best among permutation invariant procedures.

As the following example shows, there is no such procedure unless the class $\mathcal{P}$ is quite restricted or the loss function is very unrealistic.

6.2. Example. $\mathcal{P}$ contains the following two matrices P and P' and their respective orbits, i.e., all matrices P^σ, P'^σ obtained from P, P' by permutation of the players

$$P = \begin{pmatrix} 0.5 & 0.9 & 0.9 & 0.9 & 1 \\ 0.1 & 0.5 & 0.8 & 0.8 & 1 \\ 0.1 & 0.2 & 0.5 & 0.7 & 1 \\ 0.1 & 0.2 & 0.3 & 0.5 & 0.6 \\ 0 & 0 & 0 & 0.4 & 0.5 \end{pmatrix}$$

$P' = (p'_{ij})$ where $p'_{ij} = 1/(1 + e^{-(\theta_i - \theta_j)})$ with fixed $\theta_1 > \theta_2 > \cdots > \theta_5$. As in the reduced ranking problem, we take the loss function

$$\begin{aligned} L(d, P) &= 0 \qquad \text{if } d \text{ correct,} \\ &= 1 \qquad \text{otherwise.} \end{aligned}$$

Both matrices P and P' are such that the true ranking is $d = (1, 2, \cdots, 5)$ [whichever of the suggested principles under (2.2) is used to define "true ranking"].

Observe that if we restrict our attention to a single one of the two orbits, we are exactly within the setup of the "reduced ranking problem." In order to prove that there is no uniformly best ranking procedure for $\mathcal{P}$ we have to show that the reduced ranking procedure leads to different results on the two orbits or more specifically, we have to exhibit a matrix A which on the orbit of P leads to a different ranking than on the orbit of P'. For example, this matrix can be chosen as follows:

$$A = \begin{pmatrix} 0.5 & 0 & 1 & 1 & 1 \\ 1 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 1 \\ 0 & 1 & 1 & 0.5 & 0 \\ 0 & 1 & 0 & 1 & 0.5 \end{pmatrix}$$

According to the "reduced ranking procedure" we find that the best choices for the ranking vector are as follows:

On the orbit of P the best choice is any of the following three vectors: (4, 5, 1, 3, 2), (4, 5, 2, 1, 3), (4, 5, 3, 2, 1) [this means that player number one is always placed fourth (see (3.1)].

On the orbit of P' the best choice is any of the following six vectors: (1, 5, 2, 3, 4), (1, 5, 2, 4, 3), (1, 5, 3, 2, 4), (1, 5, 4, 2, 3), (1, 5, 3, 4, 2), (1, 5, 4, 3, 2) (i.e., player number one is always placed first) which clearly shows that the optimal ranking procedure on each orbit leads to different results.

7. Optimum properties of the usual ranking procedure.

7.1. Since there is no uniformly best procedure for all classes $\mathcal{P}$ (see Example (6.2) in the previous section) it is natural to ask the question, what is the class $\mathcal{P}_u$ for which the usual ranking procedure by scores $s_i = \sum a_{ij}$ turns out to be uniformly best? The following two theorems answer this question.

THEOREM 1. *Assume that P is a probability matrix such that $p_{ij} \neq 0$ for all i, j. Then $W_P(A)$ depends on A only through the score vector $s = (s_1, s_2, \cdots, s_n)$ if and only if $P = (p_{ij})$ is of the form*

$$p_{ij} = F(\theta_i - \theta_j) \tag{7}$$

where $\{\theta_i\}$ are constants and $F(t) = 1/(1 + e^{-t})$ (logistic function).

PROOF. (a) Let $p_{ij} = 1/(1 + e^{-(\theta_i - \theta_j)})$, then

$$c_{ij} = \log (p_{ij}/p_{ji}) = \theta_i - \theta_j$$

and

$$\tfrac{1}{2} \sum_{i,j} a_{ij}c_{ij} = \tfrac{1}{2} \sum_{i,j} a_{ij}(\theta_i - \theta_j) = \sum_i s_i\theta_i - \tfrac{1}{2} \sum_i \theta_i,$$

hence by (6)

$$W_P(A) = c(\theta) \exp\{\textstyle\sum s_i\theta_i\}. \tag{8}$$

Thus, s is sufficient for θ. This result can also be inferred from formula (1) of Bradley and Terry [1], since their model $p_{ij} = \pi_i/(\pi_i + \pi_j)$ corresponds to the above with $\theta_i = \log \pi_i$.

(b) To show the converse we observe that

$$c_{ij} = \theta_i - \theta_j \Leftrightarrow p_{ij} = 1/(1 + e^{-(\theta_i - \theta_j)}). \tag{9}$$

We therefore have to show that if $P(A)$ depends on A only through s, then the c_{ij} must be of the form $\theta_i - \theta_j$, or equivalently, must satisfy $c_{lk} + c_{kr} + c_{rl} = 0$ for all triplets (l, k, r). (For then, one may find such θ's by putting $\theta_1 = 0$ and $\theta_i = c_{i1}$, $i \neq 1$.)

For the proof a contrario suppose that there exists a triplet, e.g., (l, k, r) with

$$c_{lk} + c_{kr} + c_{rl} \neq 0. \tag{10}$$

On the other hand, look at the following two matrices A and A': A with $a_{lk} = a_{kr} = a_{rl} = 1$ and all other elements arbitrary but, of course, satisfying $a_{ij} + a_{ji} = 1$.

$$A' \quad \text{with} \quad a'_{lk} = a'_{kr} = a'_{rl} = 0$$
$$a'_{kl} = a'_{rk} = a'_{lr} = 1$$

and all other elements the same as in A.

One then easily computes $W_P(A) \neq W_P(A')$ because of (10). Since, obviously, $s_i = s'_i$ for all i, W_P depends on A not only through the scores but in a more complicated fashion.

7.2. This theorem implies that, if we want to find the class $\mathcal{P}_u$, for which the usual ranking method is uniformly best, we cannot hope that this class will contain more matrices than those of the form (7) (class $\mathcal{P}_l$). In fact we have the following

COROLLARY 1. *Let $\mathcal{P}$ be a class of matrices for which $p_{ij} \neq 0$ for all i, j and* *which, with each element P, also contains all matrices P^σ and let $L(d, P) = 0$ if d is the correct ranking, $= 1$ otherwise. Then, if $\mathcal{P}$ contains at least one element $P_0 \varepsilon \mathcal{P}_l$, the usual ranking procedure by scores $s_i = \sum_j a_{ij}$ is not uniformly best for $\mathcal{P}$ among invariant procedures.*

Proof. Take $P_0 \varepsilon \mathcal{P}_l$ and denote by $\mathcal{P}_0$ the orbit of P_0. By assumption $\mathcal{P}_0 \subset \mathcal{P}$ and, without loss of generality, let us assume that the vector $(1, 2, 3, \cdots, n)$ is the correct ranking for P_0. On the other hand, we know by relation (10) that for P_0 and some triplet (l, k, r)

$$c_{lk} + c_{kr} + c_{rl} \neq 0. \tag{10}$$

Consider now a matrix A with the following properties:

(i) $a_{lk} = a_{kr} = a_{rl} = 1$,

(ii) $a_{kj} = a_{rj}$ for all $j \neq l, k, r$,

(iii) the ones and zeros are distributed among the other games such that, if we rank by scores, player l may be assigned to the lth place, player k to the kth place and player r to the rth place.

Remark. Such a matrix always exists. Its construction is particularly simple if the number of players is odd, $n = 2m + 1$. Then define a matrix A' by

$$\begin{aligned} \text{for } i < j: \quad a'_{ij} &= 1 \qquad \text{if } i + j \text{ odd} \\ &= 0 \qquad \text{if } i + j \text{ even}, \\ \text{for } i > j: \quad a'_{ij} &= 1 - a'_{ji}. \end{aligned}$$

We have

$$\begin{aligned} &a'_{12} = a'_{23} = a'_{31} = 1, \\ &a'_{1j} = a'_{3j}, \qquad \text{for } j \neq 1, 2, 3, \\ &s_i = \sum_{j \neq i} a_{ij} = m \qquad \text{for all } i, \end{aligned}$$

hence, ranking by scores, any player may be assigned to any place. Then take a permutation σ for which $\sigma(l) = 2$, $\sigma(k) = 3$, $\sigma(r) = 1$ and define $A = A'^\sigma$, which gives a matrix with the properties (i), (ii), (iii).

A similar, but slightly more complicated construction works if n is even, and is left to the reader.

Provided the usual ranking procedure is uniformly best for $\mathcal{P}$ [and hence for $\mathcal{P}_0$, where consequently it has to yield a solution of the reduced ranking problem (4)] we have

$$W_{P_0^\sigma}(A) = \max \text{ on } \mathcal{P}_0 \tag{11}$$

for all those permutations σ with the property that $(\sigma(1), \cdots, \sigma(n))$ is a possible ranking vector by the usual ranking procedure.

Because of its special form, A will have in particular two permutations σ and σ', satisfying (11) and having the property $\sigma(l) = l$, $\sigma(k) = k$, $\sigma(r) = r$, $\sigma'(l) =$

l, $\sigma'(k) = r$, $\sigma'(r) = k$, $\sigma'(i) = \sigma(i)$ for $i \neq l, k, r$. Consequently

$$W_{P_0^\sigma}(A) = W_{P_0^{\sigma'}}(A) > 0 \tag{12}$$

(since $p_{ij} \neq 0$ for all i and j). But by (6′)

$$\begin{aligned} &\log W_{P_0^\sigma}(A) - \log W_{P_0^{\sigma'}}(A) \\ &\quad = \tfrac{1}{2}(c_{lk} + c_{kr} + c_{rl}) - \tfrac{1}{2}(c_{kl} + c_{rk} + c_{lr}) = c_{lk} + c_{kr} + c_{rl} \neq 0, \end{aligned} \tag{13}$$

which contradicts (12).

7.3. After having seen that essentially (i.e., if only strictly positive probability matrices are under consideration) $\mathcal{P}_l \supset \mathcal{P}_u$ we now want to prove the reverse inclusion $\mathcal{P}_u \supset \mathcal{P}_l$ which, for all "reasonable" loss functions is in fact guaranteed by the following theorem.

THEOREM 2. *Assume that* (i) $\mathcal{P} = \mathcal{P}_l$, (ii) $L(d, P) = L(d, \theta)$ *is invariant under permutations and* $L(d, \theta) \leqq L(d', \theta)$ *whenever* $d_k < d_l$, $d_k' = d_l$, $d_l' = d_k$, $d_i' = d_i$ *for the remaining* i *and* $\theta_k \geqq \theta_l$. (*This means that the loss does not decrease if two players are "wrongly interchanged" in the ranking.*) *Then the usual ranking procedure by scores* $s_i = \sum a_{ij}$ *is for* $\mathcal{P}$ *uniformly best among all permutation invariant procedures.*

PROOF.

(a) Let $\varphi_d(A)$ be a randomized ranking procedure invariant under permutations and $D = \{d\}$ the set of all possible ranking vectors. Obviously

$$\sum_{d \varepsilon D} \varphi_d(A) = 1, \qquad \text{for all } A, \tag{14}$$

and the risk function is computed as

$$R(\varphi_d, \theta) = \sum_A \sum_d L(d, \theta)\varphi_d(A) W_\theta(A) \tag{15}$$

where $W_\theta(A) = W_P(A)$ as in the previous notation (5.1).

(b) Instead of minimizing $R(\varphi_d, \theta)$ among all permutation invariant procedures it is easier to minimize the Bayes risk $\bar{R}(\varphi_d, \theta)$ with respect to the uniform a priori distribution over all vectors obtained from θ by permutation of its components

$$\bar{R}(\varphi_d, \theta) = (1/n!) \sum_\sigma \sum_A \sum_d L(d, \theta^\sigma)\varphi_d(A) W_{\theta^\sigma}(A) \tag{16}$$

where $\theta^\sigma = (\theta_1^\sigma, \theta_2^\sigma, \cdots, \theta_n^\sigma) = (\theta_{\sigma(1)}, \theta_{\sigma(2)}, \cdots, \theta_{\sigma(n)})$. Write this as

$$\bar{R}(\varphi_d, \theta) = \sum_A [\sum_d \varphi_d(A) q_d(A)] \tag{17}$$

with

$$q_d(A) = (1/n!) \sum_\sigma L(d, \theta^\sigma) W_{\theta^\sigma}(A).$$

Optimum procedures are therefore characterized by

$$\varphi_d(A) = 0 \quad \text{if } d \text{ such that } q_d(A) > \inf_{d'} q_{d'}(A). \tag{18}$$

(c) It remains to be shown that the minimum of $q_d(A)$ is attained if d is the ranking in descending order of the s_i. More precisely, we shall show that $q_d(A) \leqq q_{d'}(A)$ if $s_k \geqq s_l$, $d_k < d_l$, $d' = d^\gamma$ where $d^\gamma = (d_1^\gamma, d_2^\gamma, \cdots, d_n^\gamma) = (d_{\gamma(1)}, d_{\gamma(2)}, \cdots, d_{\gamma(n)})$ and where the permutation γ interchanges k and l and leaves all other integers at the same places.

We find

$$(19)\qquad \begin{aligned} q_d(A) - q_{d'}(A) &= (1/n!) \sum_\sigma [L(d, \theta^\sigma) W_{\theta^\sigma}(A) - L(d^\gamma, \theta^\sigma) W_{\theta^\sigma}(A)] \\ &= (1/n!) \sum_\sigma [L(d, \theta^{\sigma\gamma}) W_{\theta^{\sigma\gamma}}(A) - L(d^\gamma, \theta^{\sigma\gamma}) W_{\theta^{\sigma\gamma}}(A)]. \end{aligned}$$

Using the invariance of the loss function: $L(d, \theta) = L(d^\tau, \theta^\tau)$ we obtain from the last equation (observe that $\gamma = \gamma^{-1}$),

$$(20)\qquad q_d(A) - q_{d'}(A) = (1/n!) \sum_\sigma L(d^\gamma, \theta^\sigma) W_{\theta^{\sigma\gamma}}(A) - L(d, \theta^\sigma) W_{\theta^{\sigma\gamma}}(A)$$

and combining (19) and (20) we have

$$(21)\qquad \begin{aligned} &q_d(A) - q_{d'}(A) \\ &\qquad = (1/2n!) \sum_\sigma [L(d, \theta^\sigma) - L(d', \theta^\sigma)]\, [W_{\theta^\sigma}(A) - W_{\theta^{\sigma\gamma}}(A)] \end{aligned}$$

Writing ζ for θ^σ we have

$$\begin{aligned} W_\zeta(A) - W_{\zeta^\gamma}(A) &= c(\theta)\, [\exp\{\sum_i s_i \zeta_i\} - \exp\{\sum s_i \zeta_i^\gamma\}] \\ &= c(\theta)\, [\exp\{s_k\zeta_k + s_l\zeta_l\} - \exp\{s_k\zeta_l + s_l\zeta_k\}] \exp\{\sum_{i \neq k, l} s_i\zeta_i\}. \end{aligned}$$

For $s_k \geqq s_l$ we therefore have

$$(22)\qquad W_\zeta(A) - W_{\zeta^\gamma}(A) \leqq 0 \quad \text{if} \quad \zeta_k \leqq \zeta_l, \quad \geqq 0 \quad \text{if} \quad \zeta_l \leqq \zeta_k.$$

Our loss function $L(d, \zeta)$ has been defined (see assumption (ii) of the theorem) such that

$$(23)\qquad L(d, \zeta) - L(d', \zeta) \geqq 0 \quad \text{if} \quad \zeta_k \leqq \zeta_l, \quad \leqq 0 \quad \text{if} \quad \zeta_l \leqq \zeta_k.$$

Combining (21), (22) and (23) we conclude $q_d(A) - q_{d'}(A) \leqq 0$ if $s_k \geqq s_l$ and d, $d' = d^\gamma$ as given at the beginning of (c).

8. Final remarks.

8.1. The two theorems in Section 7 can easily be extended as follows:

(a) They remain true if a fixed number $k \geqq 1$ of games is played between each pair of players, and if a_{ij} is the number of games that player i has won against j.

(b) Theorem 1 remains true also (with obvious modifications) if each player does not play against every other one.

8.2. For more intricate generalizations of these results we refer the reader to the following paper [2].

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